J. Fluid Mech. (1969), *vol.* 36, *part* 1, *pp.* 145–159
Printed in Great Britain

Boundary-value problems in the kinetic theory of gases Part I. Slip flow

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(Received 13 May 1968)

A new method for treating boundary-value problems in gas-kinetic theory has been developed. The new method has the advantage of reproducing the bulk or asymptotic flow properties accurately whilst giving a realistic description of the behaviour of the molecular distribution function in the neighbourhood of a wall. As an example, the Kramers, or slip-flow, problem is solved for a general specular-diffuse boundary condition and some new expressions for the slip coefficient, flow speed and molecular distribution function at the surface are derived.

A brief discussion of the eigenvalue spectrum of the associated Boltzmann equation is given and its physical significance pointed out.

Certain analogies between this problem and the Milne problem in neutron transport theory are demonstrated.

1. Introduction

During the past few years there has been a marked increase in interest in certain basic problems of the kinetic theory of gases. For example, such well-posed problems as the heat transport between parallel plates, Couette flow, Poiseuille flow and slip flow have been studied in some detail. In each of these problems it is possible, under certain restrictive conditions on flow velocity and temperature difference, to linearize the Boltzmann equation and thereby subject it to the well-known techniques of linear analysis.

Before a complete solution of the above problems may be obtained it is usually found necessary to make certain assumptions about the energy exchange process between atoms of the gas. This usually takes the form of replacing the true scattering kernel by a synthetic function which preserves certain physical characteristics of the original. The well-known Bhatnagar, Gross & Krook (1954) model (B.G.K.) is an example of this technique, and also the improvements to the B.G.K. model of Gross & Jackson (1959), Cercignani (1966) and Loyalka & Ferziger (1967). The advantage of such synthetic kernels is that they enable, in many cases, analytic solutions of the Boltzmann equation to be obtained. An important disadvantage, however, is that the hydrodynamic equations are not reproduced accurately, in the sense that the coefficients of conductivity and viscosity are incorrect. This is an important limitation since in many cases the mass flow and heat transport are governed mainly by these parameters.

We are faced, therefore, with an apparent dilemma: either we sacrifice bulk flow characteristics to obtain an accurate picture of the behaviour of the gas near boundaries or we use an approximate method of solving the transport equation and thereby lose the detailed form of the solution in those very regions which are of considerable practical importance.

The present paper offers a way out of this impasse and is based on an analogy with a method developed by Williams (1968) in connexion with related neutron transport problems. It has also since been discovered that Simons (1967*a, b*) has developed a similar technique in connexion with Poiseuille flow. The present series of papers will, however, go considerably beyond the work of Simons and will treat a variety of problems.

The basic idea of the present method is to divide the solution of the Boltzmann equation into two parts: the asymptotic or hydrodynamic term and the boundary transient or Knudsen layer. The hydrodynamic part of the solution accounts for the bulk flow and heat transport and is treated exactly, in the sense that it obeys the Chapman–Enskog equations with the appropriate collision operator. The remaining term, which describes the Knudsen layer, also satisfies separately the Boltzmann equation (with modified boundary conditions), but in this equation we make use of the B.G.K. approximation. The net effect of this hybrid procedure is to maintain the correct hydrodynamic solution but to render the complete problem exactly soluble, such has not been the case in previous treatments of this problem.

The equation for the Knudsen layer, although containing an approximate representation of the energy transfer properties of the atoms of the gas, is expected to be a good approximation to the true situation. This is because the behaviour of the gas in the neighbourhood of a wall is dependent on a correct representation of the mean free path and the boundary conditions, rather than on the detailed energy exchange properties of the gas. Our model will therefore account for bulk properties exactly and will treat the boundary region in a realistic manner.

Practical application of the problems discussed above may be found in the design of vacuum equipment and more recently in the assessment of the drag characteristics of satellites in the upper atmosphere. The value of an accurate transport theory analysis of rarefied gas problems also lies in the fact that reliable slip boundary conditions for the hydrodynamic equations may be derived.

This is the first of a series of papers on these problems and in it we shall concentrate on the slip flow or Kramers problem (Kramers 1949) and consider the other problems mentioned above in later reports. Finally, it should be mentioned that the Kramers problem has already been treated in various approximations by Wang Chang & Uhlenbeck (1956), Shen (1965), Cercignani & Tironi (1966), Cercignani (1962, 1966) and Loyalka & Ferziger (1967).

The last two authors have in fact made use of the idea of subtraction of the hydrodynamic solution but have solved for the transient solution by a variational method.

2. The basic equation and boundary conditions

We consider an infinite space in which gas is flowing in the z-direction with a mass velocity $\bar{c}_z$ proportional to x, the co-ordinate perpendicular to mass flow. A plate, with accommodation coefficient β is introduced in the plane $x = 0$ and the problem is to find the stationary distribution function of the gas atoms in the half-space $x \geqslant 0$. In many respects this problem resembles the classical Milne problem of neutron transport and radiative transfer (Davison 1957; Chandrasekhar 1960) and we shall point out the similarities in due course.

The Boltzmann equation describing this situation may be written

$$c_x \frac{\partial f(\mathbf{c}, x)}{\partial x} = nJ(f, f_1), \tag{1}$$

where $f(\mathbf{c}, x)$ is the distribution function, $\mathbf{c}$ the molecular velocity normalized to $(2kT/M)^{\frac{1}{2}}$ (T is temperature and M mass) and J is the collision operator.

Now we write
$$f(\mathbf{c}, x) = f_0(\mathbf{c}, x)(1 + h(\mathbf{c}, x)), \tag{2}$$
where
$$f_0(\mathbf{c}, x) = n(m/2\pi kT)^{\frac{3}{2}} e^{-c^2} \tag{3}$$
and
$$c^2 = c_x^2 + c_y^2 + (c_z - K_0 x)^2. \tag{4}$$

In (4) K_0 is a constant representing the gradient of the velocity in the x-direction far from the plate. It is readily verified that there are no variations in density or temperature in this problem (Cercignani 1966).

Inserting (2) into (1) and neglecting terms of order h^2 and $h\,\partial f_0/\partial x$, we obtain

$$2K_0 c_x c_z + c_x(\partial h/\partial x) = nJ(h), \tag{5}$$

where, to the same order of approximation, we may set $c^2 = c_x^2 + c_y^2 + c_z^2$.

In general, the collision term $J(h)$ cannot be simplified further. However, in this work we shall be concerned with scattering models for which $nJ(h)$ can be written

$$nJ(h) = -V(c)\,h(\mathbf{c}, x) + \int d\mathbf{c}'\, e^{-c'^2} K(\mathbf{c}, \mathbf{c}')\, h(\mathbf{c}', x), \tag{6}$$

where $V(c)$ is the collision frequency for atoms of speed c and $K(\mathbf{c}, \mathbf{c}')$ determines the rate at which atoms with velocity $\mathbf{c}'$ before a collision have velocity $\mathbf{c}$ after a collision. Equation (6) is typical of the hard-sphere model.

If we now refer to a polar co-ordinate system in which (c_x, c_y, c_z) is replaced by (c, θ, χ) (see figure 1), then (5) becomes

$$2K_0 c^2 \mu(1-\mu^2)^{\frac{1}{2}} \cos\chi + c\mu \frac{\partial h(c, \mu, \chi, x)}{\partial x} + V(c)\,h(c, \mu, \chi, x)$$
$$= \int_0^\infty dc'\, c'^2 e^{-c'^2} \int_{-1}^{1} d\mu' \int_0^{2\pi} d\chi'\, K(c, c', \mu, \mu', \chi, \chi')\, h(c', \mu', \chi', x), \tag{7}$$

where
$$\mu = \cos\theta.$$

Expanding the kernel K in the Legendre polynomials as follows:

$$K(c, c', \mu, \mu', \chi, \chi') = \sum_{l=0}^{\infty} \frac{2l+1}{4\pi} K_l(c, c') \left\{ P_l(\mu)\, P_l(\mu') + 2 \sum_{m=1}^{l} \frac{(l-m)!}{(l+m)!} P_l^m(\mu)\, P_l^m(\mu') \cos m(\chi - \chi') \right\},$$

inserting into (7), multiplying by $\cos\chi$ and integrating over $\chi(0, 2\pi)$, leads to

$$2K_0 c^2\mu(1-\mu^2)^{\frac{1}{2}} + c\mu\frac{\partial g(c,\mu,x)}{\partial x} + V(c)\,g(c,\mu,x)$$
$$= \sum_{l=1}^{\infty}\frac{(2l+1)}{2l(l+1)}P_l^{(1)}(\mu)\int_0^{\infty} dc'c'^2 e^{-c'^2}K_l(c,c')\int_{-1}^{1} d\mu' P_l^{(1)}(\mu')\,g(c',\mu',x), \tag{8}$$

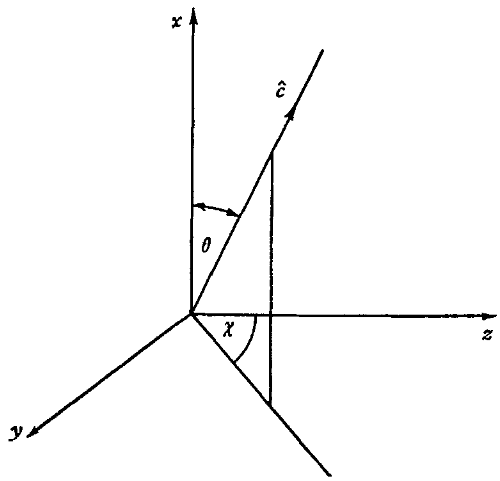

FIGURE 1. Angular co-ordinates of molecular motion. $\hat{c}$ is a unit vector in the direction of motion of a molecule.

where
$$g(c,\mu,x) = \frac{1}{\pi}\int_0^{2\pi} d\chi\cos\chi h(c,\mu,\chi,x). \tag{9}$$

Associated with (7) are the boundary conditions obeyed by the distribution function f at $x = 0$. These may be stated quite generally as (Keller 1948)

$$c_x f(\mathbf{c},0) = -\int_{c'_x<0} d\mathbf{c}' c'_x\,\Gamma(\mathbf{c},\mathbf{c}')f(\mathbf{c}',0) \tag{10}$$

for $c_x > 0$. Implicit in this condition is the fact that particles are conserved at the wall. $\Gamma(\mathbf{c},\mathbf{c}')$ is the wall-scattering function and measures the probability that an atom incident on the wall with velocity $\mathbf{c}'$ will leave it with velocity $\mathbf{c}$. Considerable experimental effort has been put into measuring Γ, but for the purpose of the present work we shall be content with the assumption of an arbitrary ratio of elastic diffuse scattering to specular reflexion. The boundary condition may then be written as

$$h(c,\mu,\chi,0) = \frac{2}{\pi}\beta\int_0^1 d\mu'\mu'\int_0^{2\pi} d\chi'\int_0^{\infty} dc'c'^3 e^{-c'^2}h(c',-\mu',\chi',0) + (1-\beta)\,h(c,-\mu,\chi,0) \tag{11}$$

for $\mu > 0$ and $0 \leqslant \chi \leqslant 2\pi$. β is the proportion of elastic diffuse scattering.

In terms of g, (11) becomes

$$g(c,\mu,0) = (1-\beta)\,g(c,-\mu,0) \quad (\mu > 0). \tag{12}$$

It is interesting to note that the term involving diffuse scattering disappears entirely. However, had we chosen a more complex angular distribution of scattered particles, this term would still, in general, be present. For the very special case of purely diffuse scattering we have the extremely simple boundary condition

$$g(c,\mu,0) = 0 \quad (\mu > 0). \tag{13}$$

We must now solve (8) subject to the boundary condition (12). Before doing this, however, let us define some quantities of practical interest.

3. The pressure tensor and mean flow velocity

The component P_{xz} of the pressure tensor is defined as follows:

$$P_{xz} = m\left(\frac{2kT}{m}\right)^{\frac{5}{2}} \int d\mathbf{c}\, c_x(c_z - K_0 x) f_0(\mathbf{c},x)\{1 + h(\mathbf{c},x)\}, \tag{14}$$

which to the same order as the equation for h may be written

$$P_{xz} = \frac{2kT}{\sqrt{\pi}} \int_0^\infty dc\, c^4 e^{-c^2} \int_{-1}^1 d\mu\, \mu(1-\mu^2)^{\frac{1}{2}} g(c,\mu,x). \tag{15}$$

If now (8) is multiplied by $c_z e^{-c^2}$ and integrated over all velocity space, we find that it reduces to

$$\frac{d}{dx}\int_0^\infty dc\, c^4 e^{-c^2} \int_{-1}^1 d\mu\, \mu(1-\mu^2)^{\frac{1}{2}} g(c,\mu,x) = 0$$

and hence

$$P_{xz} = \text{constant}. \tag{16}$$

The value of this constant will be found later.

Another quantity of interest is the value of the mean flow velocity in the z-direction as function of x, i.e. we require

$$\overline{c_z(x)} = \frac{\int d\mathbf{c}\, c_z f(\mathbf{c},x)}{\int d\mathbf{c} f(\mathbf{c},x)}. \tag{17}$$

Inserting (2) and expanding to order K, we find

$$\overline{c_z(x)} = K_0 x + \frac{1}{\sqrt{\pi}} \int_0^\infty dc\, c^3 e^{-c^2} \int_{-1}^1 d\mu\, (1-\mu^2)^{\frac{1}{2}} g(c,\mu,x). \tag{18}$$

4. The asymptotic distribution

It is well known that in the absence of boundaries the molecular distribution function in a gas with slowly varying physical parameters obeys the Chapman–Enskog equations (Chapman & Cowling 1960). In view of this fact it may be assumed that, for $x \gg 0$, $g(c,\mu,x)$ will also obey these equations, the effect of the boundary entering only in a minor fashion. Let us assume therefore that we may write $g(c,\mu,x)$ as follows:

$$g(c,\mu,x) = A(c)\, c(1-\mu^2)^{\frac{1}{2}} - 2K_0 c^2 b(c)\, \mu(1-\mu^2)^{\frac{1}{2}} - \rho(c,\mu,x), \tag{19}$$

a form which may be inferred from the nature of the inhomogeneous term in (8).

$A(c)$ and $b(c)$ are functions to be determined and $\rho(c,\mu,x)$ is a spatially transient term which decays rapidly to zero as we move away from the surface $x = 0$.

Inserting (19) into (8) and recalling the definition of $P_l^{(1)}(\mu)$, we find that A and b obey the following equations:

$$cV(c)A(c) = \int_0^\infty dc'c'^3 e^{-c'^2} K_1(c,c')A(c') \tag{20}$$

and

$$-c^2 + V(c)c^2b(c) = \int_0^\infty dc'c'^4 e^{-c'^2} K_2(c,c')b(c'). \tag{21}$$

Since $K_1(c,c')$ is related to $V(c)$ by

$$cV(c) = \int_0^\infty dc'c'^3 e^{-c'^2} K_1(c,c'), \tag{21a}$$

the solution of (20) is simply $A(c) = \text{constant} = A_0$. $b(c)$ satisfies the well-known Chapman–Enskog viscosity equation whose relation to the viscosity coefficient is given below.

It is clear that insertion of (19) into (15) leads to

$$P_{xz} = -\frac{16kT}{15\sqrt{\pi}}K_0\int_0^\infty dc\,c^6 e^{-c^2}b(c), \tag{22}$$

where in view of (16) the integral over $\rho(c,\mu,x)$ must be zero.

Recalling that K_0 is the asymptotic velocity gradient $\partial\bar{c}_z/\partial x$, we can immediately identify the coefficient of viscosity μ_v through its definition, viz.

$$P_{xz} = -\mu_v K_0, \tag{23}$$

where, in terms of absolute units of velocity, we have

$$\mu_v = \frac{8}{15}\left(\frac{2mkT}{\pi}\right)^{\frac{1}{2}}\int_0^\infty dcc^6 e^{-c^2}b(c). \tag{24}$$

With a solution of (21), (24) constitutes an exact expression for μ_v. We see therefore that the proposed solution, i.e. (8), has, at least, the correct asymptotic value.

Now we insert (19) into the expression for the mean flow velocity (18). The result is written

$$\overline{c_z(x)} = K_0x + \tfrac{1}{2}A_0 - \frac{1}{\sqrt{\pi}}\int_0^\infty dc\,c^3 e^{-c^2}\int_{-1}^1 d\mu(1-\mu^2)^{\frac{1}{2}}\rho(c,\mu,x), \tag{25}$$

where we can also define the asymptotic mean flow velocity as

$$\bar{c}_{\text{asy}}(x) = K_0x + \tfrac{1}{2}A_0. \tag{26}$$

This is the value assumed by $\overline{c_z(x)}$ at some distance from the boundary. It is through (26) that we define the 'slip coefficient' ζ, viz.

$$\bar{c}'_{\text{asy}}(0)\,\zeta = \bar{c}_{\text{asy}}(0). \tag{27}$$

Equation (27) may be used as a boundary condition for the hydrodynamical equations and is directly analogous to 'asymptotic neutron transport theory' (Davison 1957) in which ζ is called the linear extrapolation distance.

From (26) we see that

$$\zeta = A_0/2K_0 \tag{28}$$

and our next task is, therefore, to obtain A_0.

5. The transient term

After substituting (19) into (8) we arrive at the following equation for $\rho(c,\mu,x)$:

$$\left[c\mu\frac{\partial}{\partial x}+V(c)\right]\rho(c,\mu,x) = \sum_{l=1}^{\infty}\frac{(2l+1)}{2l(l+1)}P_l^{(1)}(\mu)\int_0^\infty dc'c'^2e^{-c'^2}K_l(c,c')\int_{-1}^{1}d\mu'P_l^{(1)}(\mu')\rho(c',\mu',x) \quad (29)$$

subject to the boundary condition

$$\rho(c,\mu,0) = (1-\beta)\rho(c,-\mu,0)+\beta A_0c(1-\mu^2)^{\frac{1}{2}}-(2-\beta)\,2K_0c^2b(c)\mu(1-\mu^2)^{\frac{1}{2}}, \quad (30)$$

where $0 \leqslant \mu \leqslant 1.$

Equation (29) cannot be solved in closed form as it stands. However, if we approximate the scattering kernel on the right-hand side by a suitable synthetic function, then an analytic solution by the method of Wiener & Hopf is possible (Noble 1957). Thus, we set

$$K_l(c,c') = \gamma cc'V(c)\,V(c')\,\delta_{l,1}, \quad (31)$$

where $$\gamma^{-1} = \int_0^\infty dcc^4e^{-c^2}V(c).$$

This approximation amounts to a neglect of the higher orders of anisotropic scattering and replaces $K_1(c,c')$ by a separable, or degenerate, kernel. The great advantage of this particular choice is that it preserves the collision rate or mean free path, in the sense that the new K_1 still obeys (21*a*); indeed it is defined to do this. We justify this approximation by arguing that, in the neighbourhood of the boundary, collisions with the wall and direct free flight effects are more important than interparticle collisions. The free flight, or transport mean free path behaviour, is governed mainly by $V(c)$, which is reproduced exactly by our model. The boundary condition at the wall involves no approximation whatsoever.

With (31), the equation for ρ reads

$$[c\mu(\partial/\partial x)+V(c)]\rho(c,\mu,x) = \tfrac{3}{4}\gamma(1-\mu^2)^{\frac{1}{2}}cV(c)\,\Phi(x), \quad (32)$$

where $$\Phi(x) = \int_0^\infty dc'c'^3e^{-c'^2}V(c')\int_{-1}^{1}d\mu'(1-\mu'^2)^{\frac{1}{2}}\rho(c',\mu',x). \quad (33)$$

Equation (32) constitutes a mixed boundary-value problem which is to be solved by the Wiener–Hopf technique.

We define the Laplace transform

$$\tilde{\Phi}(s) = \int_0^\infty e^{-sx}\Phi(x)\,dx$$

which, in view of the behaviour of ρ as $x\to\infty$, exists for $\mathrm{Re}\,(s) > -\Sigma_{\min}$, where $\Sigma_{\min}$ equals the minimum value of $V(c)/c$. For the hard-sphere model it may be shown that

$$\left[\frac{V(c)}{c}\right]_{\min} = \lim_{c\to\infty}\left(\frac{V(c)}{c}\right) \equiv \lim_{c\to\infty}\Sigma(c) = \Sigma(\infty) = \Sigma_{\min}$$

and we shall in our analysis assume that this is generally true for all molecular scattering laws.

Applying the transform to (32) leads to

$$-c\mu\rho(c,\mu,0)+[V+c\mu s]\tilde{\rho}(c,\mu,s) = \tfrac{3}{4}\gamma(1-\mu^2)^{\frac{1}{2}}\,c\,V(c)\,\tilde{\Phi}(s), \tag{34}$$

where

$$\tilde{\rho}(c,\mu,s) = \int_0^\infty e^{-sx}\rho(c,\mu,x)\,dx.$$

Now, dividing (34) by $V+c\mu s$, multiplying by the factor $c^3V(c)e^{-c^2}(1-\mu^2)^{\frac{1}{2}}$ and integrating over $c(0,\infty)$ and $\mu(-1,1)$, we obtain

$$g_+(s)+g_-(s) = \tilde{K}(s)\,\tilde{\Phi}(s), \tag{35}$$

where

$$\tilde{K}(s) = 1-\tfrac{3}{4}\gamma\int_0^\infty dc\,c^4V^2(c)\,e^{-c^2}\int_{-1}^{1}\frac{d\mu(1-\mu^2)}{V+c\mu s}, \tag{36}$$

$$g_+(s) = \int_0^\infty dc\,c^4V(c)\,e^{-c^2}\int_0^1\frac{d\mu\,\mu(1-\mu^2)^{\frac{1}{2}}\rho^*(c,\mu,0)}{V+c\mu s} \tag{37}$$

and

$$g_-(s) = \int_0^\infty dc\,c^4V(c)\,e^{-c^2}\int_{-1}^0\frac{d\mu\,\mu(1-\mu^2)^{\frac{1}{2}}\rho(c,\mu,0)}{V+c\mu s}. \tag{38}$$

In (37) we have denoted by ρ^* the terms on the right-hand side of (30).

As mentioned earlier, we will apply the Wiener–Hopf technique to solve (38). The basis of this method depends upon the possibility of rearranging both sides of the equation so that each is analytic in a half-plane; these half-planes overlap one another, i.e. there must be a common region of analyticity.

We note below the regions in which the various terms in (35) are analytic:

$$\begin{aligned}
\tilde{\Phi}(s)&: \quad \operatorname{Re}(s) > -\Sigma_{\min},\\
\tilde{K}(s)&: \quad -\Sigma_{\min} < \operatorname{Re}(s) < \Sigma_{\min},\\
g_+(s)&: \quad \operatorname{Re}(s) > -\Sigma_{\min},\\
g_-(s)&: \quad \operatorname{Re}(s) < \Sigma_{\min}.
\end{aligned}$$

Clearly the required conditions for analyticity are not satisfied. We therefore employ the conventional Wiener–Hopf factorization technique and define a function $\tau(s)$ by

$$\tau(s) = \frac{s^2-\Sigma_{\min}^2}{s^2}\tilde{K}(s). \tag{39}$$

Since $\tilde{K}(s)$ has a double zero at $s = 0$, the function $\tilde{K}(s)/s^2$ is entirely free from zeros in the strip $-\Sigma_{\min} < \operatorname{Re}(s) < \Sigma_{\min}$. Furthermore the factor $s^2-\Sigma_{\min}^2$ ensures that $\tau(s) \to 1$ as $|s| \to \infty$, thus the logarithm of τ tends to zero as $|s| \to \infty$. We may therefore define two new functions $\tau_\pm(s)$ as follows:

$$\ln\tau(s) = \ln\tau_+(s)-\ln\tau_-(s) \tag{40}$$

or

$$\tau(s) = \frac{\tau_+(s)}{\tau_-(s)},$$

where, by Cauchy's theorem,

$$\ln\tau_\pm(s) = \frac{1}{2\pi i}\int_{\pm\beta'-i\infty}^{\pm\beta'+i\infty}\ln\tau(u)\frac{du}{u-s}. \tag{41}$$

$\tau_+(s)$ is analytic in the half-plane $\operatorname{Re}(s) < \beta'$ and $\tau_-(s)$ in the half-plane $\operatorname{Re}(s) > -\beta'$, where $-\Sigma_{\min} < -\beta' < \operatorname{Re}(s) < \beta' < \Sigma_{\min}$.

Substituting for $\bar{K}(s)$ from (39) and (40), (35) may be rewritten

$$g_+(s)\frac{s-\Sigma_{\min}}{\tau_+(s)}+g_-(s)\frac{s-\Sigma_{\min}}{\tau_+(s)} = \frac{s^2}{s+\Sigma_{\min}}\frac{\tilde{\Phi}(s)}{\tau_-(s)}. \tag{42}$$

The right-hand side of (42) is now analytic in the half-plane $\text{Re}(s) > -\beta'$ and the second term on the left-hand side is analytic in the half-plane $\text{Re}(s) < \beta'$. The first term on the left-hand side remains analytic only in the strip

$$-\Sigma_{\min} < \text{Re}(s) < \beta'.$$

We now write $g_+(s)/\tau_+(s)$ as a Cauchy integral, viz.

$$\frac{g_+(s)}{\tau_+(s)} = G^+(s)-G^-(s), \tag{43}$$

where
$$G^{\pm}(s) = \frac{1}{2\pi i}\int_{\pm\beta'-i\infty}^{\pm\beta'+i\infty}\frac{g_+(u)}{\tau_+(u)}\frac{du}{u-s}. \tag{44}$$

$G^+(s)$ is analytic in the half-plane $\text{Re}(s) < \beta$ and $G^-(s)$ in the half-plane $\text{Re}(s) > -\beta'$. This decomposition is possible since $g_+(u)/\tau_+(u) \sim O(u^{-1})$ and hence the integrals exist.

Finally, then, (42) becomes

$$(s-\Sigma_{\min})\,G^+(s)+g_-(s)\frac{s-\Sigma_{\min}}{\tau_+(s)} = \frac{s^2\tilde{\Phi}(s)}{(s-\Sigma_{\min})\,\tau_-(s)}+(s-\Sigma_{\min})\,G^-(s). \tag{45}$$

Each side is now analytic in a half-plane and there is a common region of analyticity. We may conclude, therefore, by Liouville's theorem, that the functions on the right- and left-hand sides are analytic continuations of one another. Examination of the behaviour of (45) as $|s| \to \infty$ indicates that we may set

$$\frac{s^2\tilde{\Phi}(s)}{(s+\Sigma_{\min})\,\tau_-(s)}+(s-\Sigma_{\min})\,G^-(s) = C_0, \tag{46}$$

where C_0 is a constant.

We obtain C_0 by imposing the condition that $\Phi(x)$ goes to zero at least as fast as $\exp(-\Sigma_{\min}x)$. This is equivalent to setting the coefficients of s^{-1} and s^{-2} to zero in an asymptotic expansion of $\tilde{\Phi}(s)$ in powers of s^{-1}. Thus we find $C_0 = -\Sigma_{\min}G^-(0)$, and in addition $G^-(0) = \Sigma_{\min}G^{-\prime}(0)$.

To evaluate $G^-(s)$ it is only necessary to invert the orders of the u and (c,μ) integrations in (44). We then obtain

$$G^-(s) = -\int_0^\infty dc\,c^3V(c)\,e^{-c^2}\int_0^1 d\mu\,\frac{\mu(1-\mu^2)^{\frac{1}{2}}\,\tau_-(\Sigma(c)/\mu)}{\Sigma(c)+s\mu}\,\rho^*(c,\mu,0). \tag{47}$$

ρ^* contains the unknown functions $A_0/2K_0$ and $\rho(c,-\mu,0)$. Thus using the condition derived above that $G^-(0) = \Sigma_{\min}G^{-\prime}(0)$ we find from (47) that

$$\frac{A_0}{2K} = \frac{3\gamma\Sigma_{\min}\tau_-(0)}{4\beta}(2-\beta)\int_0^\infty dc\,c^6b(c)\,e^{-c^2}\int_0^1 d\mu\,\tau_-\left(\frac{\Sigma(c)}{\mu}\right)\mu^2(1-\mu^2)\,(1+\mu\Sigma_{\min}l(c))$$
$$-\frac{3\gamma\Sigma_{\min}\tau_-(0)}{4\beta}\frac{1-\beta}{2K_0}\int_0^\infty dc\,c^4\,e^{-c^2}\int_0^1 d\mu\,\tau_-\left(\frac{\Sigma(c)}{\mu}\right)\mu(1-\mu^2)^{\frac{1}{2}}\,(1+\mu\Sigma_{\min}l(c))\,\rho(c,-\mu,0), \tag{48}$$

where we have used a relationship involving the function $\tau_-(s)$ which is given in appendix A.

$A_0/2K_0$ is by definition the slip coefficient, but before it can be evaluated it will be necessary to obtain the equation for $\rho(c, -\mu, 0)$.

Integration of (32) leads to the relation

$$\rho(c, -\mu, 0) = \frac{3\gamma}{4}\frac{(1-\mu^2)^{\frac{1}{2}}}{\mu} V(c)\,\hat{\Phi}\left(\frac{\Sigma(c)}{\mu}\right) \tag{49}$$

for $\mu > 0$.

If we now insert the value of C_0 into (46) and then use (49), the equation for $\rho(c, -\mu, 0)$ becomes

$$\begin{aligned}\frac{1}{2K_0}\rho(c, -\mu, 0) &= \tfrac{3}{4}\gamma(1-\mu^2)^{\frac{1}{2}} c(\Sigma(c)+\mu\Sigma_{\min})\,\tau_-\left(\frac{\Sigma(c)}{\mu}\right)\\ &\times\int_0^\infty dc'\,c'^4 e^{-c'^2}\int_0^1 d\mu'\,\mu'(1-\mu'^2)^{\frac{1}{2}}\tau_-\left(\frac{\Sigma(c')}{\mu'}\right)\frac{\Sigma(c')+\mu'\Sigma_{\min}}{\mu\Sigma(c')+\mu'\Sigma(c)}\\ &\times\left\{\frac{1-\beta}{2K_0}\rho(c', -\mu', 0)+\beta\frac{A_0}{2K_0}c'(1-\mu'^2)^{\frac{1}{2}}-(2-\beta)c'^2 b(c')\,\mu'(1-\mu'^2)^{\frac{1}{2}}\right\}.\end{aligned} \tag{50}$$

Thus, together, (48) and (50) yield the slip coefficient ζ and the quantity $\rho(c, -\mu, 0)/2K_0$. To obtain the value of the velocity at the surface, $\bar{c}_z(0)$, it is only necessary to insert $\rho(c, -\mu, 0)$ in (25).

For the spatial behaviour of the molecular distribution function it is convenient to return to (35). Then we find

$$\Phi(x) = \frac{1}{2\pi i}\int_C \frac{\{g_+(s)+g_-(s)\}}{K(s)} e^{sx}\,ds, \tag{51}$$

where C is a line lying to the right of all singularities of the integrand. Knowing $\Phi(x)$, the complete distribution function may be reconstructed by integrating the first order equation (32).

The complete inversion of (51) is represented by a rather complex function. The general form of the solution, however, may be written as

$$\Phi(x) = \int_{\Sigma_{\min}}^\infty A(u)\,e^{-ux}\,du. \tag{52}$$

Thus $\overline{c_z(x)}$ becomes

$$\frac{1}{K_0}\overline{c_z(x)} = x+\zeta-\int_{\Sigma_{\min}}^\infty B(u)\,e^{-ux}\,du. \tag{53}$$

The form of this solution is sketched in figure 2 and we note that it consists of two well-defined parts: the asymptotic $(x+\zeta)$ and a spatial transient. In practice, the magnitude of the spatial transient is of some importance for it governs the applicability of the conventional hydrodynamic equations to problems of this type. The form of the transient given by (53) indicates that it should be negligible at about one maximum mean free path from the interface (i.e. $\Sigma_{\min}^{-1}$). Calculations for the analogous Milne problem in neutron transport theory show this to be true. This suggests, therefore, that we may use the hydrodynamic equations with the appropriate slip boundary condition in regions which are no nearer than $\Sigma_{\min}^{-1}$ from a surface. Before accepting this as a fact, however, it should be borne in mind that we have approximated the scattering kernel and have thereby altered the form of the transient solution. It is also worth noting that the constant collision rate approximation implies $\Sigma_{\min} = 0$. In this case the transient may be important up to a substantial distance from the boundary. However, numerical work by Loyalka & Ferziger (1967) has shown that this effect is not very serious.

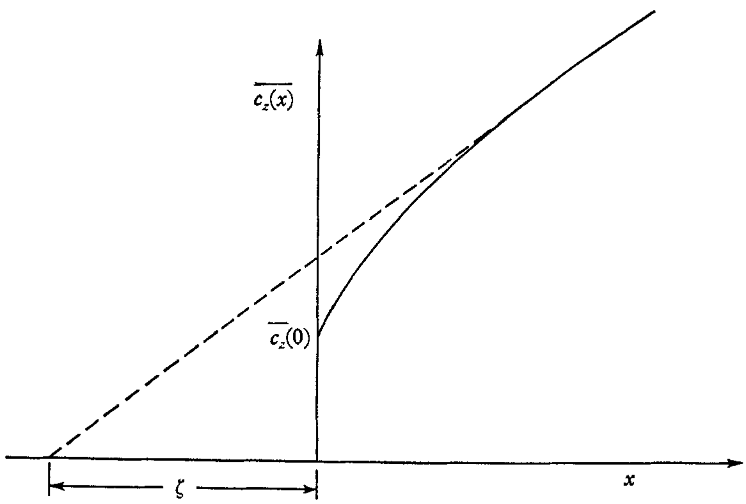

FIGURE 2. Mean flow velocity in z-direction as a function of distance x from the boundary. ζ is the slip coefficient.

A more fundamental study of the Boltzmann equation (Williams 1966) shows that, in general, the equation for $g(c, \mu, x)$ admits solutions of the form $g_\nu(c, \mu)\, e^{-\nu x}$ and that the spatial behaviour of g depends upon the eigenvalues ν. For our simple model it was shown that only a double, discrete eigenvalue at $\nu = 0$ existed, together with a continuous spectrum extending from $\nu = \Sigma_{\min}$ to $\nu = \infty$. The double zero gives rise to the asymptotic part of the solution and the continuum to the integral term. In practice, there may exist other discrete eigenvalues $\nu_j\,(j = 1, 2, \ldots)$ which would contribute additional terms of the form $A_j \exp(-\nu_j x)$ to (53). These ν_j are less than $\Sigma_{\min}$ and hence their contribution to the non-asymptotic (i.e. non-hydrodynamic) part of the solution will persist over a larger distance from the boundary. Preliminary work on this problem indicates that the ν_j are possibly absent altogether, but if they do exist they will be close to $\Sigma_{\min}$ in absolute value. Thus the above criterion for the validity of the hydrodynamic approximation is still a good one.

6. Diffuse reflexion

In the case of diffuse reflexion, $\beta = 1$, and the expressions for ζ and $g(c, -\mu, 0)$ may be obtained in a closed form. ζ is simply the first term of (48) with $\beta = 1$ and g may be found from (50) and (19). The result is

$$(1/2K_0)\, g(c, -\mu, 0) = c^2 b(c)\, \mu(1-\mu^2)^{\frac{1}{2}} + \tfrac{3}{4}\gamma(1-\mu^2)^{\frac{1}{2}}\, c\psi(c,\mu) \int_0^\infty dc'\, c'^6 b(c')\, e^{-c'^2} \int_0^1 d\mu'\, \mu'^2 \frac{(1-\mu'^2)\,\psi(c',\mu')}{\mu\Sigma(c')+\mu'\Sigma(c)}, \tag{54}$$

where
$$\psi(c,\mu) = (\Sigma(c)+\mu\Sigma_{\min})\, \tau_-(\Sigma(c)/\mu). \tag{55}$$

Similarly,

$$\frac{1}{2K_0}\bar{c}_z(0) = \frac{1}{4\sqrt{\pi}}\int_0^\infty dc\, c^5 e^{-c^2} b(c) + \frac{3\gamma}{4\sqrt{\pi}} \int_0^\infty dc\, c^4 e^{-c^2} \int_0^1 d\mu (1-\mu^2)\, \psi(c,\mu) \times \int_0^\infty dc'\, c'^6 e^{-c'^2} b(c') \int_0^1 d\mu' (1-\mu'^2) \frac{\mu'^2 \psi(c',\mu')}{\mu\Sigma(c')+\mu'\Sigma(c)}. \tag{56}$$

These expressions are generalizations of those given by previous authors and are clearly more accurate.

A special case of the above work which is of some interest is the complete B.G.K. approximation, i.e. when we apply the approximation given by (31) to the asymptotic distribution as well. This case has been dealt with by Cercignani (1966). An immediate shortcoming of this approximation may be seen in the value for the viscosity. Using the hard-sphere model, we find from (21) and (22) that the exact value of μ_v is given by Pekeris & Alterman (1957)

$$(\mu_v/\mu_1) = 1{\cdot}016034,$$

whereas for the B.G.K. model, which corresponds to setting $K_2 = 0$ in (21), we find

$$(\mu_v/\mu_1) = 0{\cdot}630862$$

with
$$\mu_1 = \frac{5(\pi m k T)^{\frac{1}{2}}}{16}.$$

Thus an error of about 40 % in the bulk flow properties arises from the B.G.K. approximation.

One of the virtues of the B.G.K. model is that extremely simple expressions are available for ζ, $g(c, -\mu, 0)$ and $\bar{c}_z(0)$. This is useful since these exact results can be used to check the accuracy of less accurate but more flexible methods of solving the transport equation, e.g. the variational and discrete ordinate techniques.

We find, using some properties of the $\psi(c,\mu)$ functions given in the appendix, that

$$\zeta = \frac{1}{\Sigma_{\min}} + \frac{\tau'_-(0)}{\tau_-(0)} \tag{57}$$

$$= \frac{2}{\pi}\int_0^\infty dt \frac{I_1(t)}{I_2(t)}, \tag{57a}$$

where $$I_1 = \int_0^\infty dc\, c^8 V(c)\, e^{-c^2} \int_0^1 d\mu \frac{\mu^4(1-\mu^2)}{(V^2+c^2\mu^2t^2)^2},$$
$$I_2 = \int_0^\infty dc\, c^6 V(c)\, e^{-c^2} \int_0^1 d\mu \frac{\mu^2(1-\mu^2)}{V^2+c^2\mu^2t^2}.$$

The μ-integrals are easily evaluated but the result will not be given here owing to its length. Two particular cases are, however, worth quoting:

(i) $V(c) = \lambda =$ constant.

$$\zeta\lambda = \frac{2}{\pi}\int_0^\infty dt \left\{ \int_0^\infty \frac{dx\, x^4 e^{-x^2}}{(1+x^2t^2)^2} \Big/ \int_0^\infty \frac{dx\, x^2 e^{-x^2}}{1+x^2t^2} \right\}. \tag{58}$$

(ii) $V(c) = c$. $$\zeta = \frac{1}{\pi}\int_0^\infty \frac{dt}{t^2} \left\{ \frac{\frac{4}{3}t^2+5-(5+3t^2)(1/t)\tan^{-1}t}{\frac{2}{3}t^2+1-(1+t^2)(1/t)\tan^{-1}t} \right\}. \tag{59}$$

The second case corresponds to a constant interaction cross-section between atoms, whereas the first assumes a constant collision rate.

In terms of the average mean free path $\bar{l}$, where

$$\bar{l} = \frac{\int_0^\infty dc\, c^3 e^{-c^2} l(c)\, dc}{\int_0^\infty dc\, c^3 e^{-c^2}\, dc}, \quad \text{with} \quad l(c) = \Sigma(c)^{-1}, \tag{60}$$

we find for case (i): $\zeta/\bar{l} = 0{\cdot}7644$, case (ii): $\zeta/\bar{l} = 0{\cdot}5825$, and for the true hard-sphere collision rate as given in appendix B, $\zeta/\bar{l} = 0{\cdot}6510$.

These values compare with those deduced from the variational method of Loyalka & Ferziger as follows: case (i), 0·7578; case (ii), 0·5792; and for hard-spheres, 0·6439. This is a striking example of the power of the variational method.

If the value of ζ obtained by the new method presented in this paper (i.e. (48) with $\beta = 1$) is compared with the value deduced by the B.G.K. model, i.e. (48) but with $b(c) = V(c)^{-1}$, then it may be inferred that the new value of ζ is about 40 % greater than the B.G.K. value. We arrive at this conclusion by analogy with the calculation of the viscosity coefficient which, it is recalled, also depends on an average over $b(c)$. These conclusions are again confirmed by the numerical work of Loyalka & Ferziger.

Another quantity of interest which may be obtained in a fairly simple manner is the angular distribution of particles impinging on the plate, viz.

$$\frac{1}{2K_0} g(c, -\mu, 0) = \frac{c(1-\mu^2)^{\frac{1}{2}}}{\Sigma_{\min}\Sigma(c)\,\tau_-(0)} \psi(c,\mu), \tag{61}$$

where $\tau_-(0)$ is given in the appendix.

Cercignani (1966) has given a simple value of $\bar{c}_z(0)$, but we believe this to be erroneous since it does not seem possible to reduce the expression further than the following quadrature:†

$$\frac{1}{K_0}\bar{c}_z(0) = \frac{2}{\sqrt{\pi}}\frac{1}{\Sigma_{\min}\tau_-(0)} \int_0^\infty dc\, c^4 e^{-c^2} l(c) \int_0^1 d\mu (1-\mu^2)\, \psi(c,\mu). \tag{62}$$

† This error has been confirmed by Prof. Cercignani (private communication to the author, 1968).

For the special case of $V(c) = \lambda$, we find $\bar{c}_z(0) = 4K_0\bar{l}/3\sqrt{(2\pi)}$, and for $V = c$, $\bar{c}_z(0) = K_0\bar{l}/\sqrt{5}$, which agrees with Cercignani's formula.

7. Summary and conclusions

A new method for treating boundary-value problems in the kinetic theory of gases has been developed which has the advantage of reproducing the asymptotic flow properties accurately and giving a realistic description of the molecular distribution function in the neighbourhood of a boundary.

For the slip flow problem with a general diffuse-specular boundary condition we have obtained expressions for the slip coefficient, speed of flow at the boundary and distribution of molecules incident on the plate. These results are more accurate than any yet derived and provide useful boundary conditions for use with the hydrodynamic equations when these are valid.

The main uncertainty in the present method lies in the use of the B.G.K. approximation for the transient or boundary-layer effect. This model predicts a physically acceptable eigenvalue spectrum for the spatial operator of the Boltzmann equation, but may lack certain refinements associated with a more realistic energy exchange mechanism. This problem is under investigation and results will be presented soon. In the meantime we are using the present technique to study other flow problems, viz. Couette and Poiseuille flow, and also heat transport between parallel plates.

The author is indebted to J. Spain for his capable programming of the integrals.

Appendix A

The following properties of the function $\psi(c,\mu)$ are of use in simplifying some of the equations in the text. With $\psi(c,\mu) = (\Sigma(c)+\mu\Sigma_{\min})\,\tau_-(\Sigma(c)/\mu)$ we find:

$$\frac{3\gamma}{4}\int_0^\infty dc'\,c'^5\,e^{-c'^2}\int_0^1 d\mu'\,\mu'(1-\mu'^2)\,\frac{\psi(c',\mu')}{\mu\Sigma(c')+\mu'\Sigma(c)} = \frac{1}{\psi(c,\mu)}, \tag{A 1}$$

$$\frac{3\gamma}{4}\int_0^\infty dc\,c^5\,e^{-c^2}l(c)\int_0^1 d\mu\,\mu(1-\mu^2)\,\psi(c,\mu) = \frac{1}{\Sigma_{\min}\tau_-(0)}, \tag{A 2}$$

$$\frac{3\gamma}{4}\int_0^\infty dc\,c^5\,e^{-c^2}l(c)\int_0^1 d\mu\,\mu^2(1-\mu^2)\,\psi(c,\mu) = \frac{1}{\Sigma_{\min}\tau_-(0)}\left\{\frac{1}{\Sigma_{\min}}+\frac{\tau'_-(0)}{\tau_-(0)}\right\}, \tag{A 3}$$

where a prime indicates differentiation.

$\psi(c,\mu)$ is closely related to a function arising in the study of the Milne problem in neutron transport theory (Williams 1964) and is the velocity dependent equivalent of Chandrasekhar's H-function (Chandrasekhar 1960).

Equation (A 1) may be derived by applying the Wiener–Hopf technique to the following equation:

$$g(s) = K(s)\,\tilde{\Phi}(s), \tag{A 4}$$

where $\tilde{\Phi}$ and K are defined in the text and

$$g(s) = \int_0^\infty dc\,c^4\Sigma(c)\,e^{-c^2}\int_{-1}^0 d\mu\,\frac{\mu(1-\mu^2)^{\frac{1}{2}}\Psi(c,\mu,0)}{\Sigma(c)+s\mu}, \tag{A 5}$$

with $$\Psi(c, -\mu, 0) = \frac{3\gamma}{4}(1-\mu^2)^{\frac{1}{2}} c\Sigma(c)\,\hat{\Phi}\left(\frac{\Sigma(c)}{\mu}\right). \tag{A 6}$$

$\tau_-(0)$ is derived from (41), from which we find

$$\tau_-(0) = \frac{1}{\sqrt{\tau(0)}} = \frac{\sqrt{5}}{\Sigma_{\min}}\left\{\frac{\int_0^\infty dc\, c^5\Sigma(c)\, e^{-c^2}}{\int_0^\infty dc\, c^5 l(c)\, e^{-c^2}}\right\}^{\frac{1}{2}} \tag{A 7}$$

Similarly $\tau'_-(0)/\tau_-(0)$ may be shown to be given by

$$\frac{\tau'_-(0)}{\tau_-(0)} = \frac{1}{\pi}\int_0^\infty \frac{\tau'(it)}{\tau(it)}\frac{dt}{t}. \tag{A 8}$$

Insertion of $\tau(u)$ leads to (57 a) of the text.

Appendix B

For the hard-sphere model, the collision frequency is given by

$$V(c) = c\Sigma(c) = \frac{c}{l(c)} = \frac{e^{-c^2}}{\sqrt{\pi}} + \left(c + \frac{1}{2c}\right)\operatorname{erf}(c),$$

where the 'cross-section' $n\pi\sigma^2$ is set equal to unity.

The average value of $l(c)$, as defined by (60) of the text, is equal to 0·742893.

REFERENCES

Bhatnagar, P. L., Gross, E. P. & Krook, M. 1954 *Phys. Rev.* **94**, 5.

Cercignani, C. 1962 *Ann. Phys.* **20**, 219.

Cercignani, C. 1966 *Ann. Phys.* **40**, 469.

Cercignani, C. & Tironi, G. 1966 *Il Nuovo Cim.* **43**, 64.

Chandrasekhar, S. 1960 *Radiative Transfer*. New York: Dover.

Chapman, S. & Cowling, T. G. 1960 *The Mathematical Theory of Non-Uniform Gases.* Cambridge University Press.

Davison, B. 1957 *Neutron Transport Theory.* Oxford University Press.

Gross, E. P. & Jackson, E. A. 1959 *Phys. Fluids*, **2**, 432.

Keller, J. B. 1948 *Comm. Appl. Math.* **1**, 275.

Kramers, H. A. 1949 *Nuovo Cim.* **6** Suppl., 887.

Loyalka, S. K. & Ferziger, J. H. 1967 *Phys. Fluids*, **10**, 1833.

Noble, B. 1957 *The Wiener–Hopf Technique.* Oxford: Pergamon.

Pekeris, C. L. & Alterman, Z. 1957 *Proc. Nat. Acad. Sci., U.S.A.* **43**, 998.

Shen, S. F. 1965 *5th International Symposium on Rarefied Gas Dynamics*, p. 623. Edited by C. L. Brundin. Academic Press.

Simons, S. 1967 *a* *Proc. Roy. Soc.* A **301**, 387.

Simons, S. 1967 *b* *Proc. Roy. Soc.* A **301**, 401.

Wang Chang, C. S. & Uhlenbeck, G. E. 1956 *Engng. Res. Inst. University of Michigan* Report Non r-122, 4-(15).

Williams, M. M. R. 1964 *Nucl. Sci. Engng.* **18**, 260.

Williams, M. M. R. 1966 *The Slowing Down and Thermalization of Neutrons.* Amsterdam: North-Holland.

Williams, M. M. R. 1968 *J. Nucl. Energy*, **22**, 535.